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Jordan KdV Systems and Painlevé Property

Ayşe (Kalkanlı) Karasu[1]

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The Painlevé property of Jordan KdV systems in two dimensions is studied. It is shown that a subclass of these equations on a nonassociative algebra possesses the Painlevé property.

1. INTRODUCTION

Svinolupov (1991) introduced many-field Korteweg–de Vries (KdV) equations

$$u_t^i = u_{xxx}^i + a_{jk}^i u^j u_x^k, \qquad i = 1, \dots, N \tag{1.1}$$

where u^i depend on variables x and t, and a_{jk}^i is a set of constants symmetric with respect to the subscripts. He showed that there is a one-to-one correspondence between such equations and Jordan algebras. Specifically, Jordan KdV systems have an infinite algebra of generalized symmetries, an infinite series of local conservation laws, and a recursion operator. The systems corresponding to simple Jordan algebras are called irreducible. In two dimensions all the systems related to two-dimensional Jordan algebras contain a scalar KdV equation and a linear equation which are not really coupled (Svinolupov, 1991, 1994).

In this work we consider the system (1.1) for $N = 2$. We apply the Painlevé test for partial differential equations introduced by Weiss *et al.* (1983) to the system of coupled KdV equations without *a priori* assumptions about the algebraic nature of the system. We find the sets of constants a_{jk}^i for which the system (1.1) possesses the Painlevé property. A subclass of these equations on a nonassociative algebra has the Painlevé property. By

[1] Department of Physics, Middle East Technical University, 06531 Ankara, Turkey.

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using the truncated expansions of the solutions we also obtain the auto-Bäcklund transformations for these equations.

2. SYMMETRY APPROACH

The recursion operator for the scalar Korteweg–de Vries equation ($N = 1$) is given in Olver (1993) as

$$L = D^2 + \tfrac{2}{3}u + \tfrac{1}{3}u_x D^{-1} \tag{2.1}$$

where $D \equiv d/dx$. An analogous operator also exists for Jordan systems. In Svinolupov (1991) it is stated as a theorem that any Jordan system (1.1) is integrable and possesses a formal recursion operator:

$$\begin{aligned} L = {} & D^2 + \tfrac{2}{3}a(i)u^i + \tfrac{1}{3}u^i_x D^{-1}a(i) \\ & + \tfrac{1}{9}u^j D^{-1}u^k D^{-1}[a^n_{jk}a(n) - a(k)a(j)] \end{aligned} \tag{2.2}$$

where the matrices $a(j)$ are determined by the formula $(a(j))^i_k = a^i_{jk}$ and the constants a^i_{jk} are the structure constants of a Jordan algebra satisfying the identities

$$\begin{aligned} & a^n_{jk}(a^i_{nr}a^r_{ms} - a^i_{mr}a^r_{ns}) + a^n_{km}(a^i_{nr}a^r_{js} - a^i_{jr}a^r_{ns}) \\ & + a^n_{mj}(a^i_{nr}a^r_{ks} - a^i_{kr}a^r_{ns}) = 0 \end{aligned} \tag{2.3}$$

We consider a system of two nonlinear equations of the form

$$\begin{aligned} u_t &= u_{xxx} + c_1 uu_x + c_2(uv_x + vu_x) + c_3 vv_x \\ v_t &= v_{xxx} + d_1 uu_x + d_2(uv_x + vu_x) + d_3 vv_x \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} & c_1 = a^1_{11}, && d_1 = a^2_{11}, && u = u^1 \\ & c_2 = a^1_{21} = a^1_{12}, && d_2 = a^2_{12} = a^2_{21}, && v = u^2 \\ & c_3 = a^1_{22}, && d_3 = a^2_{22}, && \end{aligned} \tag{2.5}$$

The recursion operator (2.2) for this problem can be expressed as a 2×2 matrix whose components are

$$\begin{aligned} (L)_{11} = {} & D^2 + \tfrac{2}{3}(c_1 u + c_2 v) + \tfrac{1}{3}(u_x D^{-1}c_1 + v_x D^{-1}c_2) \\ & - \tfrac{1}{9}(uD^{-1})(vD^{-1})F_1 - \tfrac{1}{9}(vD^{-1})^2 F_2 \end{aligned} \tag{2.6}$$

$$\begin{aligned} (L)_{12} = {} & \tfrac{2}{3}(c_2 u + c_3 v) + \tfrac{1}{3}(u_x D^{-1}c_2 + v_x D^{-1}c_3) \\ & + \tfrac{1}{9}(uD^{-1})^2 F_1 + \tfrac{1}{9}(vD^{-1})(uD^{-1})F_2 \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
(L)_{21} &= \tfrac{2}{3}(d_1 u + d_2 v) + \tfrac{1}{3}(u_x D^{-1} d_1 + v_x D^{-1} d_2) \\
&\quad - \tfrac{1}{9}(uD^{-1})(vD^{-1})F_3 + \tfrac{1}{9}(vD^{-1})^2 F_1 \qquad (2.8) \\
(L)_{22} &= D^2 + \tfrac{2}{3}(d_2 u + d_3 v) + \tfrac{1}{3}(u_x D^{-1} d_2 + v_x D^{-1} d_3) \\
&\quad + \tfrac{1}{9}(uD^{-1})^2 F_3 - \tfrac{1}{9}(vD^{-1})(uD^{-1})F_1 \qquad (2.9)
\end{aligned}
$$

where

$$
F_1 = c_3 d_1 - c_2 d_2, \qquad F_2 = c_2^2 - c_1 c_3 + c_3 d_2 - c_2 d_3 \qquad (2.10)
$$
$$
F_3 = d_1 d_3 - d_2^2 + c_1 d_2 - c_2 d_1
$$

and the structure constants c_i and d_i satisfy the following identities:

$$
\begin{aligned}
(c_1 - 2d_2)F_1 &= 0, \qquad (c_1 - 2d_2)F_2 = 0, \qquad (c_1 - 2d_2)F_3 = 0 \\
(d_3 - 2c_2)F_1 &= 0, \qquad (d_3 - 2c_2)F_2 = 0, \qquad (d_3 - 2c_2)F_3 = 0 \qquad (2.11) \\
d_1 F_1 &= 0, \qquad d_1 F_2 = 0, \qquad d_1 F_3 = 0 \\
c_3 F_1 &= 0, \qquad c_3 F_2 = 0, \qquad c_3 F_3 = 0
\end{aligned}
$$

If F_1, F_2, F_3 vanish, the recursion operator reduces to a form similar to (2.1). This case corresponds to an associative algebra in which the system (2.4) decouples.

3. PAINLEVÉ ANALYSIS

A partial differential equation has the Painlevé property when its solutions are single-valued about the movable singularity manifold. If the singularity manifold is determined by

$$
\phi(x^0, x^1, \ldots, x^n) = 0 \qquad (3.1)
$$

and u^a ($a = 1, \ldots, N$) satisfy a system of partial differential equations (N-equations), then the Painlevé expansion is given by

$$
u^a = \phi^{\alpha_a} \sum_{k=0}^{\infty} u_k^a(x^0, x^1, \ldots, x^n)\phi^k \qquad (3.2)
$$

where u_k^a are analytic functions of $(x^0, x^1, \ldots, x^n)$ in a neighborhood of the manifold (3.1). The substitution of (3.2) into the partial differential equations under consideration determines the possible values of α_a and gives the recursion relations for u_k^a. A set of partial differential equations is said to have the Painlevé property in the sense of Weiss *et al.* provided α_a are integers, the recursion relation are consistent, and the series expansion (3.2) contains the correct number of arbitrary functions. Applying the Painlevé analysis to equations (2.4), we obtain the following:

(i) Leading order analysis. Substituting $u = u_0\phi^{\alpha_1}$ and $v = v_0\phi^{\alpha_2}$ into the leading terms of (2.4), we have $\alpha_1 = \alpha_2 = -2$ and the equations for u_0 and v_0,

$$c_1u_0^2 + 2c_2u_0v_0 + c_3v_0^2 + 12u_0\phi_x^2 = 0 \tag{3.3}$$

$$d_1u_0^2 + 2d_2u_0v_0 + d_3v_0^2 + 12v_0\phi_x^2 = 0 \tag{3.4}$$

(ii) Resonances. Substituting

$$u = u_0\phi^{-2} + \beta^1\phi^{r-2}, \qquad v = v_0\phi^{-2} + \beta^2\phi^{r-2} \tag{3.5}$$

into the leading terms of equations (2.4) and requiring that β^1 and β^2 be arbitrary, we have

$$\begin{aligned}&\{\phi_x^4(r-2)^2(r-3)^2 + \phi_x^2(r-2)(r-3)[u_0(d_2+c_1) + v_0(d_3+c_2)]\\ &\quad + [(u_0c_1 + v_0c_2)(u_0d_2 + v_0d_3) - (u_0c_2 + v_0c_3)(u_0d_1 + v_0d_2)]\}\\ &\quad \times (r-4)^2 = 0\end{aligned} \tag{3.6}$$

The roots of this equation determine the resonances. $r = 4$ is a double root which satisfies the equation identically. We must always have the root $r = -1$, since it represents the arbitrariness of the singularity manifold $\phi(x, t) = 0$. This is possible if

$$\begin{aligned}&144\phi_x^4 + 12\phi_x^2[u_0c_1 + v_0(d_3 + 2c_2)]\\ &\quad + v_0[u_0(d_3c_1 - d_1c_3) + 2v_0(d_3c_2 - d_2c_3)] = 0\end{aligned} \tag{3.7}$$

If this equation is satisfied, we have another root, $r = 6$. Using equations (3.3), (3.4), and (3.7), we find that equation (3.6) becomes

$$\begin{aligned}&12\phi_x^4(r^2 - 5r + 6) + 12\phi_x^2(u_0d_2 - v_0c_2)\\ &\quad + v_0[u_0(d_1c_3 - d_3c_1) + 2v_0(d_2c_3 - d_3c_2)] = 0\end{aligned} \tag{3.8}$$

The roots of this equation are

$$r_1 = \frac{15\phi_x^2 - \sqrt{\ }}{6\phi_x^2}, \qquad r_2 = \frac{15\phi_x^2 + \sqrt{\ }}{6\phi_x^2} \tag{3.9}$$

r_1 and r_2 must be integers, say n_1 and n_2; then we have the following values of resonances:

$$r = -1, 4, 4, 6, n_1, n_2 \qquad \text{where} \quad n_1 + n_2 = 5 \tag{3.10}$$

Now let us examine the different cases for n_1, n_2.

Case 1. Let $n_1 = 0$, $n_2 = 5$; then from equation (3.8) we have

$$72\phi_x^4 + 12\phi_x^2(u_0d_2 - v_0c_2) + v_0[u_0(d_1c_3 - d_3c_1) + 2v_0(d_2c_3 - d_3c_2)] = 0 \tag{3.11}$$

Equations (3.3), (3.4), (3.7), and (3.11) must be solved for u_0 and v_0. Since one of the roots is zero, the function u_0 or v_0 must be arbitrary. If u_0 is arbitrary, the equations under consideration imply $v_0 = \alpha\phi_x^2 + \beta$, where α and β are constants. Requiring that the equations for u_0 and v_0 be satisfied, we have the following solution:

$$\alpha = -12/d_3, \qquad \beta = -u_0d_2/c_2$$

$$d_1 = 0, \qquad d_2 = c_1/2, \qquad d_3 = 2c_2, \qquad c_3 = 0$$

$$u_0 \quad \text{is arbitrary,} \qquad v_0 = \frac{1}{2c_2}(-12\phi_x^2 - u_0c_1) \tag{3.12}$$

(iiia) Arbitrary functions. To discuss the arbitrariness of the functions corresponding to resonance values $-1, 0, 4, 4, 5, 6$, we have to substitute

$$u = \sum_{j=0}^{6} u_j\phi^{j-2}, \qquad v = \sum_{j=0}^{6} v_j\phi^{j-2}$$

into equations (2.4) and obtain the recursion relations for u_j and v_j. Solving these relations, we have

$$j = 0 \qquad u_0 = \text{arbitrary}; \qquad v_0 = -(12\phi_x^2 + c_1u_0)/2c_2 \tag{3.13}$$

$$j = 1 \qquad v_1 = [\phi_{xx}(12\phi_x^2 - u_0c_1) + \phi_xu_{0x}c_1]/2\phi_x^2c_2$$

$$u_1 = (\phi_{xx}u_0 - \phi_xu_{0x})/\phi_x^2 \tag{3.14}$$

$$\begin{aligned} j = 2 \qquad v_2 = {} & [8\phi_{xxx}\phi_x(-6\phi_x^2 + u_0c_1) + \phi_{xx}^2(36\phi_x^2 - 21u_0c_1) \\ & + 18\phi_{xx}\phi_xu_{0x}c_1 + 6\phi_x^2(2\phi_x\phi_t - u_{0xx}c_1) + \phi_x\phi_tu_0c_1]/24\phi_x^4c_2 \\ u_2 = {} & [-8\phi_{xxx}\phi_xu_0 + \phi_{xx}(21\phi_{xx}u_0 - 18\phi_xu_{0x}) \\ & + \phi_x(6\phi_xu_{0xx} - \phi_tu_0)]/12\phi_x^4 \end{aligned} \tag{3.15}$$

$$\begin{aligned} j = 3 \qquad v_3 = {} & [3\phi_{xt}\phi_x^2(-4\phi_x^2 + u_0c_1) + 3\phi_{xxxx}\phi_x^2(4\phi_x^2 - u_0c_1) \\ & + 4\phi_{xxx}\phi_{xx}\phi_x(-12\phi_x^2 + 7u_0c_1) - 10\phi_{xxx}\phi_x^2u_{0x}c_1 \\ & + \phi_{xx}(36\phi_{xx}^2\phi_x^2 - 39\phi_{xx}^2u_0c_1 + 33\phi_{xx}\phi_xu_{0x}c_1 \\ & + 12\phi_x^3\phi_t - 12\phi_x^2u_{0xx}c_1 - \phi_x\phi_tu_0c_1) \\ & + \phi_x^2(2\phi_xu_{0xxx}c_1 - 2\phi_xu_{0t}c_1 + \phi_tu_{0x}c_1)]/24\phi_x^6c_2 \end{aligned}$$

$$
\begin{aligned}
u_3 = {} & [-3\phi_{xt}\phi_x^2 u_0 + 3\phi_{xxxx}\phi_x^2 u_0 + \phi_{xxx}\phi_x(-28\phi_{xx}u_0 + 10\phi_x u_{0x}) \\
& + \phi_{xx}(39\phi_{xx}^2 u_0 - 33\phi_{xx}\phi_x u_{0x} + 12\phi_x^2 u_{0xx} + \phi_x\phi_t u_0) \\
& + \phi_x^2(-2\phi_x u_{0xxx} + 2\phi_x u_{0t} - \phi_t u_{0x})]/12\phi_x^6 \qquad (3.16)
\end{aligned}
$$

$j = 4$ the compatibility conditions are satisfied identically, which means u_4 and v_4 are arbitrary functions

$j = 5$ $v_5 =$ arbitrary

$$
\begin{aligned}
u_5 = {} & [-v_5\phi_x(12\phi_x^2 + u_0c_1\phi_x + 4v_0c_2) - v_{4x}(12\phi_x^2 + u_0c_1 + 4v_0c_2) \\
& - v_4(12\phi_{xx}\phi_x + u_1c_1\phi_x + 4v_1c_2\phi_x + u_{0x}c_1 + 4v_{0x}c_2) \\
& - v_0c_1u_{4x} - u_4(\phi_x v_1c_1 + v_{0x}c_1) - 6\phi_x v_{3xxx} \\
& - v_{3x}(6\phi_{xxx} + u_1c_1 + 4v_1c_2) - v_3(2\phi_{xxx} + 2\phi_x v_2c_2 \\
& + \phi_x u_2c_1 - 2\phi_t + u_{1x}c_1 + 4v_{1x}c_2) - u_{3x}v_1c_1 \\
& - u_3(\phi_x v_2c_1 + v_{1x}c_1) \\
& - 2v_{2xxx} - v_{2x}(u_2c_1 + 4v_2c_2) + 2v_{2t} - v_2c_1u_{2x}]/\phi_x v_0c_1 \qquad (3.17)
\end{aligned}
$$

$j = 6$ $u_6 =$ arbitrary,

$$
\begin{aligned}
v_6 = {} & [-2\phi_x v_0c_1u_6 - v_{5x}(u_0c_1 + 4v_0c_2 + 36\phi_x^2) \\
& - v_5(36\phi_{xx}\phi_x + 2\phi_x u_1c_1 + 8\phi_x v_1c_2 + u_{0x}c_1 + 4v_{0x}c_2) \\
& - u_{5x}v_0c_1 - u_5(2\phi_x v_1c_1 + v_{0x}c_1) - 12\phi_x v_{4xx} \\
& - v_{4x}(12\phi_{xx} + u_1c_1 + 4v_1c_2) \\
& - v_4(4\phi_{xxx} + 2\phi_x u_2c_1 + 8\phi_x v_2c_2 - 4\phi_t + u_{1x}c_1 + 4v_{1x}c_2) \\
& - u_{4x}v_1c_1 - u_4(v_{1x}c_1 + 2\phi_x v_2c_1) - 2v_{3xxx} \\
& - v_{3x}(u_2c_1 + 4v_2c_2) + 2v_{3t} - u_{3x}v_2c_1 - u_{2x}v_3c_1 \\
& - v_{2x}(u_3c_1 + 4v_3c_2) - 2\phi_x v_3(u_3c_1 \\
& + 2v_3c_2)]/[2\phi_x(24\phi_x^2 + u_0c_1 + 4v_0c_2)] \qquad (3.18)
\end{aligned}
$$

Since ϕ, u_0, u_4, v_4, v_5, u_6 are arbitrary functions corresponding to the resonances $(-1, 0, 4, 4, 5, 6)$, the system of equations (2.4) with $c_3 = d_1 = 0$, $d_3 = 2c_2$, $d_2 = c_1/2$ passes the Painlevé test. It is easy to check that the Jordan algebra is nonassociative with these values of c_i and d_i. In Weiss (1983, 1986) it was shown that the Bäcklund transformations can be obtained by truncating the expansions (3.2) at constant level terms, that is, $u_j = 0$ if $j \geq 3$, $v_j = 0$ if $j \geq 3$. This is possible if

$$u_{2t} = u_{2xxx} + c_1 u_2 u_{2x} + c_2(u_2 v_{2x} + v_2 u_{2x})$$

$$v_{2t} = v_{2xxx} + \frac{c_1}{2}(u_2 v_{2x} + v_2 u_{2x}) + 2c_2 v_2 v_{2x} \tag{3.19}$$

Equations (3.13)–(3.15) and (3.19) will be consistent if

$$\phi_t = \phi_{xxx} - \frac{3}{2}\frac{\phi_{xx}^2}{\phi_x} + \lambda\phi_x, \qquad \lambda = \text{const} \tag{3.20}$$

which can be formulated in terms of the Schwarzian derivative

$$\frac{\phi_t}{\phi_x} - \{\phi; x\} = \lambda \tag{3.21}$$

and the function u_0 must be a solution of the linear equation

$$\begin{aligned} &2\phi_x^3 u_{0t} - 2\phi_x^3 u_{0xxx} + 12\phi_{xx}\phi_x^2 u_{0xx} \\ &\quad - \phi_x(18\phi_{xx}^2 + 10\lambda\phi_x^2 - 9\phi_x\phi_t)u_{0x} \\ &\quad + 2\phi_{xx}(3\phi_{xx}^2 + 8\lambda\phi_x^2 - 9\phi_x\phi_t)u_0 = 0 \end{aligned} \tag{3.22}$$

Then,

$$\begin{aligned} u &= u_2 - \frac{1}{\phi}\left(\frac{u_0}{\phi_x}\right)_x + \frac{u_0}{\phi} \\ v &= v_2 + \frac{6}{c_2}(\ln\phi)_{xx} + \frac{c_1}{2c_2\phi}\left[\left(\frac{u_0}{\phi_x}\right)_x - \frac{u_0}{\phi}\right] \end{aligned} \tag{3.23}$$

will define the Bäcklund transformations for the Jordan KdV system, which generate nontrivial solutions from trivial ones. Note that a particular solution of (3.22) is $u_0 = C\phi_x^2$, where C is a constant.

Case 2. If $n_1 = 1$, $n_2 = 4$, the test fails, since the number of arbitrary functions is less than the number of resonances $(-1, 4, 4, 6, 1, 4)$.

Case 3. Let $n_1 = 2$, $n_2 = 3$; then from equation (3.8) we have

$$12\phi_x^2(u_0 d_2 - v_0 c_2) + v_0[u_0(d_1 c_3 - d_3 c_1) + 2v_0(d_2 c_3 - d_3 c_2)] = 0 \tag{3.24}$$

To solve equations (3.3), (3.4), (3.7), and (3.24) for u_0 and v_0, let

$$v_0 = \alpha\phi_x^2 + \beta \qquad \text{and} \qquad u_0 = \delta\phi_x^2 + \gamma$$

where α, β, γ, δ are constants. Substituting these into equations (3.3), (3.4), (3.7), and (3.24), we have

$$\begin{aligned}
&\beta = 0, \qquad \gamma = 0, \qquad \delta \neq 0 \\
&d_1 = -\frac{1}{\delta^2}(2d_2\delta\alpha + d_3\alpha^2 + 12\alpha) \\
&d_2 = -\frac{1}{\delta^2}(d_3\delta\alpha - c_2\delta\alpha - c_3\alpha^2) \\
&c_1 = -\frac{1}{\delta^2}(2c_2\delta\alpha + c_3\alpha^2 + 12\delta) \\
&u_0 = \delta\phi_x^2, \qquad v_0 = \alpha\phi_x^2
\end{aligned} \tag{3.25}$$

(iiib) Arbitrary functions. Substituting

$$u = \sum_{j=0}^{6} u_j\phi^{j-2}, \qquad v = \sum_{j=0}^{6} v_j\phi^{j-2}$$

into equations (2.4), we observe that these equations pass the Painlevé test if

$$d_3 = \frac{1}{\delta(c_2\delta + c_3\alpha)}(c_2^2\delta^2 + 3c_2c_3\delta\alpha + 2c_3^2\alpha^2 + 12c_3\delta) \tag{3.26}$$

where

$$\begin{aligned}
u_0 &= \delta\phi_x^2, \qquad v_0 = \alpha\phi_x^2 \\
u_1 &= -\delta\phi_{xx}, \qquad v_1 = -\alpha\phi_{xx} \\
v_2 &= \{u_2[\phi_x^2(c_2\delta\alpha + c_3\alpha^2 + 12\delta)] \\
&\quad + \delta^2[\phi_x(-4\phi_{xxx} + \phi_t) + 3\phi_{xx}^2]\}/[\phi_x^2\delta(c_2\delta + c_3\alpha)] \\
v_3 &= \{u_3[\phi_x^4(c_2\delta\alpha + c_3\alpha^2 + 12\delta)] + \delta^2[\phi_x^2(-2\phi_{xt} + \phi_{xxxx}) \\
&\quad + \phi_x\phi_{xx}(-4\phi_{xxx} + \phi_t) + 3\phi_{xx}^3]\}/\phi_x^4\delta(c_2\delta + c_3\alpha)
\end{aligned} \tag{3.27}$$

The expressions for u_5, v_5, and v_6 are very extensive, therefore are not presented here. The functions u_2, u_3, u_4, v_4 are arbitrary, and u_6 is also arbitrary if (3.26) is valid. But in this case F_1, F_2, F_3 in (2.11) vanish, where d_1, d_2, c_1 are given in (3.25), which implies that we have an associative algebra. Thus, in this case the system of equations (2.4) decouples.

4. CONCLUSION

We conclude that the system of equations

$$u_t = u_{xxx} + c_1 u u_x + c_2(u v_x + v u_x)$$

$$v_t = v_{xxx} + \frac{c_1}{2}(u v_x + v u_x) + 2c_2 v v_x$$

possesses the Painlevé property, has Bäcklund transformations, and corresponds to a nonassociative Jordan algebra. However, this system can be written as

$$U_t = U_{xxx} + U U_x$$

$$V_t = V_{xxx} + \tfrac{1}{2}(UV)_x$$

where $U = c_1 u + 2c_2 v$ and $V = c_1 u - 2c_2 v$. For a given solution U of the KdV equations, V is obtained by solving the linear equation. This result is consistent with that given in Svinolupov (1994).

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